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Electromagnetic Energy and Momentum from a Charged Particle

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Abstract

We compute the flux of the stress–energy tensor across a tube surrounding the world line of a charged particle. By slight modifications of the definition of the Coulomb energy-momentum, the resulting expression contains the radiation reaction term (proportional to the square of the four-acceleration) but not the Schott term (proportional to the derivative of the acceleration). The equation of motion for the particle derived from this expression implies a variable rest mass.

1. Introduction

The problem of the relativistic equation of motion of a classical charged particle has been with us now for a long time, and it is a matter of taste to decide whether a satisfactory solution has been found or not.

We restrict ourselves here to a single charged particle subject to an unspecified force, with no incoming radiation field.

The most elegant solution has been developed in the work of Dirac (1938) and Rohrlich (1965), who consider a point particle. The result is the Lorentz-Dirac equation (see footnote 1)

$$mw_\mu - (e^2/6\pi)(\dot{w}_\mu + w^2 u_\mu) = F_\mu \tag{1.1}$$

[1] We choose a somewhat different notation from that used by Rohrlich (1965). We use a time-favoring metric $g_{\mu\nu}$ (this changes the sign of the scalar products when compared to those in the above-mentioned book) and natural units such that $c = 1$ and $\epsilon_0 = 1$ (which introduces some factors of 4π). The world line is specified in terms of the proper time by $\xi_\mu(\tau)$, and u_μ is the four-velocity $d\xi_\mu/d\tau$ (not the normal to the world line, called here b_μ). We do not use v_μ to avoid confusion between $\mathbf{u} = d\boldsymbol{\xi}/d\tau$ and $\mathbf{v} = d\boldsymbol{\xi}/dt$. For the same reason, the four-acceleration $du_\mu/d\tau$ is designated by w_μ instead of a_μ. Derivatives with respect to τ are indicated by a dot, and we use the modified summation convention, $a_\mu b_\mu = a_0 b_0 - \mathbf{a}\cdot\mathbf{b}$, for repeated lower Greek indices.

Its "derivation" is based on energy-momentum balance and involves both the retarded and advanced fields from the charge, although this is not necessarily so (Teitelboim, 1970a, b; 1971a, b; Teitelboim and López, 1971). A review of the theory of the classical electron is also given by Rohrlich (1973).

Crucial to these considerations is the computation of the flux of the stress-energy tensor through a tube around a portion of the world line, which in addition to the change in the Coulomb energy-momentum supplies a term proportional to the Abraham vector

$$\Gamma_\mu = (e^2/6\pi)(\dot{w}_\mu + w^2 u_\mu) \tag{1.2}$$

The name radiation reaction for the above term, or for the first part only, is a misnomer. The determination of the radiation rate

$$\mathscr{R} = -(e^2/6\pi)w^2 \tag{1.3}$$

shows that this name is proper for the second one. The term with the derivative of the acceleration, called Schott term, is more appropriately related to the fields near the world line (Teitelboim, 1970a). This term, rightly called troublesome, changes radically the nature of the equation of motion when compared to that of a neutral particle. It is no longer sufficient to specify the initial position and velocity of the particle, but also the acceleration (or an asymptotic condition) are required to determine the motion of the particle, leading to effects such as preacceleration (Rohrlich, 1965). Although they are unobservable in practice, they are difficult to accept in this context.

There is also considerable confusion about the computation of the flux of the stress-energy tensor across the tube surrounding the world line. We show that no reference to $\dot{w}_\mu$ need appear, which results in a modified equation of motion where the Schott term is absent (Bonnor, 1974; Rowe, 1974). As is often the case in computations involving infinite quantities, slight changes in the definitions lead to significant modifications of the results. The change in the equation of motion brings with it the undesirable choice between a variable mass and a nonelectromagnetic force. The first alternative appears preferable to us, although it contradicts our experience; the solution of this difficulty would then lie in the quantization of the theory.

This leads us back to the frequently asked question: Why bother with a classical electron theory if it is known that quantum effects are important? We find that not only is it esthetically pleasing to remove a difficulty in an otherwise elegant theory, but that such a theory would be of considerable help in the formulation of quantum electrodynamics in the context of relativistic quantum mechanics (as opposed to the theory of quantized fields). In nonrelativistic quantum mechanics, the Coulomb interaction in the Hamiltonian represents the interaction between two charged point particles, not the energy of two charge distributions proportional to the probability densities; we are interested in a relativistic generalization of this Coulomb term.

We present our computations in considerable detail only to emphasize the exact nature of certain results which are often considered approximations. Our principal concern is the proper interpretation of the different terms; ours differs from that of other authors.

We introduce a somewhat different definition of the energy and momentum of the Coulomb field in Section 2, and we present a detailed computation of the flux of the stress-energy tensor across the tube surrounding the world line in Section 3. The next section contains a discussion of possible equations of motion, and we conclude with some remarks in Section 5.

2. *Energy and Momentum Associated with the Velocity Fields*

The basic assumption that we make is that Maxwell's equations are valid for the field produced by point particles. We use a Lorentz gauge, and the Liénard-Wiechert potentials are

$$A_\mu(x) = \frac{e}{4\pi} \frac{u_\mu[\tau_R(x)]}{R_\alpha(x) u_\alpha[\tau_R(x)]} \tag{2.1}$$

where

$$R_\mu(x) = x_\mu - \xi_\mu[\tau_R(x)] \tag{2.2}$$

and the retarded proper time is determined by

$$R^2 = 0, \qquad R_0 > 0 \tag{2.3}$$

that is, $\xi(\tau_R)$ is on the backward light cone from x. In the usual manner we determine the fields

$$F_{\mu\nu} = \frac{e}{4\pi}\left\{\frac{u_\mu R_\nu - u_\nu R_\mu}{(R\cdot u)^3} - \left[\frac{w_\mu R_\nu - w_\nu R_\mu}{(R\cdot u)^2} - \frac{(u_\mu R_\nu - u_\nu R_\mu) R\cdot w}{(R\cdot u)^3}\right]\right\} \tag{2.4}$$

and the symmetrized stress-energy tensor

$$\Theta_{\mu\nu} = -\frac{e^2}{16\pi^2}\left\{\frac{[(1-R\cdot w)^2 + w^2 (R\cdot u)^2] R_\mu R_\nu}{(R\cdot u)^6} - \frac{(1-R\cdot w)(u_\mu R_\nu + u_\nu R_\mu)}{(R\cdot u)^5} - \frac{w_\mu R_\nu + w_\nu R_\mu}{(R\cdot u)^4} + \frac{g_{\mu\nu}}{2(R\cdot u)^4}\right\} \tag{2.5}$$

The first term in (2.4) is the velocity field and decreases like $1/\rho^2$ for large ρ, while the other term is the acceleration or radiation field proportional to $1/\rho$, where

$$\rho = R\cdot u \tag{2.6}$$

The source of the field is the singular charge-current density,

$$j_\mu(x) = e\int d\tau\, \delta[x - \xi(\tau)]\, u_\mu(\tau) \tag{2.7}$$

The bound or Coulomb energy is usually obtained in the rest frame of a particle, and it diverges unless the computation excludes a (small) region a distance ϵ from the charge. We recall that the elementary definition of the potential energy of charged particles excludes this self-energy, which appears when an expression found for continuous charge distributions (where the self-energy is negligible) is applied to point particles.

This definition is less appropriate when the particle is accelerated, or when we are dealing with several particles in motion with respect to each other. The choice of the rest frame becomes rather arbitrary. A consideration of particle decay or pair creation also shows this definition to be in conflict with the ideas of propagation of the fields with a speed not larger than 1.

Instead, we define the bound energy-momentum in terms of the flux of the stress-energy tensor across the light cone with origin at the particle, excluding again a region of size ϵ. That is,

$$P_\mu{}^C = \int_\Sigma \Theta_{\mu\nu}\, d\sigma_\nu \tag{2.8}$$

where $\Theta_{\mu\nu}$ is given by (2.5) and Σ is shown in Figure 1. The surface element is obtained from the general expression

$$d\sigma_\mu = \pm\, \epsilon_{\mu\nu\lambda\rho}(\partial x_\nu/\partial\theta_1)(\partial x_\lambda/\partial\theta_2)(\partial x_\rho/\partial\theta_3)\, d\theta_1\, d\theta_2\, d\theta_3 \tag{2.9}$$

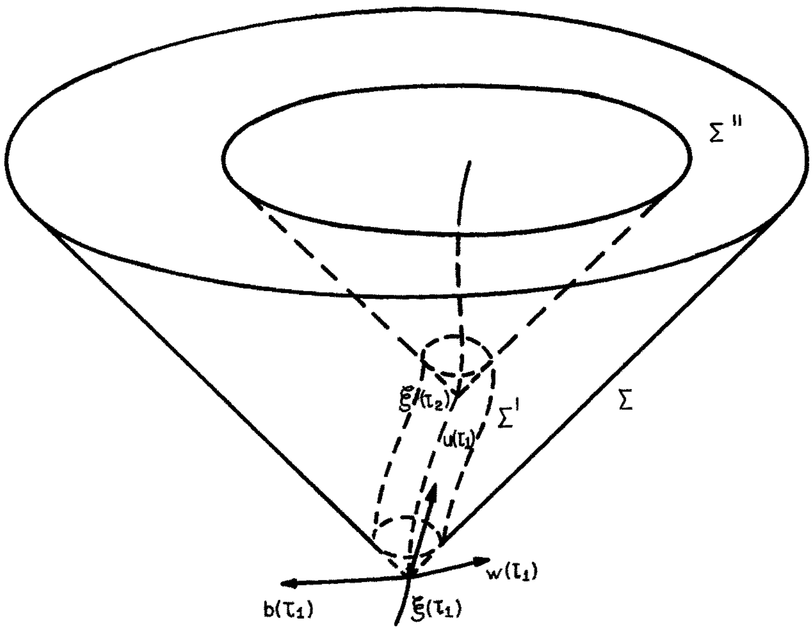

Figure 1—Diagram showing the surface Σ on the light cone, used to compute the Coulomb energy-momentum, and the surface Σ' of the tube around the world line. The surface Σ'', when removed to infinity, is used to compute the radiated energy-momentum.

where $\theta_1, \theta_2, \theta_3$ are arbitrary parameters and $\epsilon_{\mu\nu\lambda\rho}$ is the antisymmetric tensor with

$$\epsilon_{0123} = +1 \tag{2.10}$$

which is valid even for lightlike surfaces (Schild, 1960; Marx, 1975). It is unnecessary to make any reference to the "2-content" of the surface element (Synge, 1956).

The equation of the light-cone with its vertex at $\xi(\tau)$ is

$$x(\rho, \theta, \varphi) = \xi + \rho[u + b(\theta, \varphi)] \tag{2.11}$$

where θ and φ are the spherical coordinates of the unit vector $\hat{b}$ in the rest frame at τ.

The relations

$$u \cdot b = 0 \tag{2.12}$$

$$b^2 = -1 \tag{2.13}$$

imply that $\partial b/\partial\theta$ and $\partial b/\partial\varphi$ are normal to u and b, and it is easy to see that they are also normal to each other. Thus, we complete an orthonormal tetrad with unit vector b' and b'' parallel to $\partial b/\partial\theta$ and $\partial b/\partial\varphi$, respectively. Then

$$\partial x_\nu/\partial\rho = u_\nu + b_\nu \tag{2.14}$$

$$\partial x_\lambda/\partial\theta = \rho\sqrt{[-(\partial b/\partial\theta)^2]}\, b'_\lambda \tag{2.15}$$

$$\partial x_\rho/\partial\varphi = \rho\sqrt{[-(\partial b/\partial\varphi)^2]}\, b''_\rho \tag{2.16}$$

and

$$d\sigma_\mu = \rho^2(u_\mu + b_\mu)\, d\Omega\, d\rho \tag{2.17}$$

where

$$d\Omega = \sqrt{[(\partial b/\partial\theta)^2(\partial b/\partial\varphi)^2]}\, d\theta\, d\varphi \tag{2.18}$$

is the element of solid angle. Since

$$R_\mu = \rho(u_\mu + b_\mu) \tag{2.19}$$

$$u \cdot w = 0 \tag{2.20}$$

we find that

$$\Theta_{\mu\nu}\, d\sigma_\nu = (e^2/32\pi^2\rho^2)(u_\mu + b_\mu)\, d\rho\, d\Omega \tag{2.21}$$

Thus, there is no contribution from the acceleration fields; this is an exact result, not an approximation. We integrate over the solid angle and over ρ from ϵ to infinity; we get no contribution from b_μ, and the final result is

$$P_\mu{}^C(\tau) = (e^2/8\pi\epsilon)u_\mu(\tau) \tag{2.22}$$

Precisely the same result is obtained if we compute the Coulomb energy of a charged particle (outside a sphere of radius ϵ) in its rest frame, that is, over the spacelike hyperplane perpendicular to its four-velocity. But here we have added the total energy as measured by the information collecting spherical surface that moves out from the particle with the speed of light. Since the light-cone is a well-defined geometrical entity in Minkowski space, this quantity is independent of the choice of reference frame and can be included in the mass of the particle by the renormalization procedure.

3. Stress-Energy Flux Across a Tube Near the World Line

We now consider a tube around the world line between the light-cones at $\xi(\tau_1)$ and $\xi(\tau_2)$. The "radius" of the tube is the small distance ϵ, but it is measured along the light-cones instead of the normal b. The equation of this surface, shown in Figure 1, is

$$x(\tau, \theta, \varphi) = \xi(\tau) + \epsilon[u(\tau) + b(\tau, \theta, \varphi)] \tag{3.1}$$

where τ goes from τ_1 to τ_2, and θ and φ are defined as in Section 2.

A straightforward application of the divergence theorem, generalized to the four-dimensional spacetime with the Lorentz metric, shows that the flux of the stress-energy tensor across the tube has to be equal to the change in the Coulomb energy as defined above, plus the flux across the surface at infinity, the radiated energy-momentum (Rohrlich, 1965)

$$P_\mu{}^R(\tau_1, \tau_2) = -(e^2/6\pi) \int_{\tau_1}^{\tau_2} w^2 u_\mu \, d\tau \tag{3.2}$$

since the divergence of $\Theta_{\mu\nu}$ vanishes throughout the enclosed volume. There is no contribution from $\dot{w}_\mu$, so there is no need to decide whether the Schott term is related to the radiation or to the bound fields.

We verify this result by carrying out the explicit calculation in a manner similar to that in Section 2 in order to highlight possible differences with the usual expressions. The surface element is determined by

$$\partial x_\nu/\partial\tau = u_\nu + \epsilon(w_\nu + \dot{b}_\nu), \tag{3.3}$$

$$\partial x_\lambda/\partial\theta = \epsilon\sqrt{[-(\partial b/\partial\theta)^2]}\, b'_\lambda, \tag{3.4}$$

$$\partial x_\rho/\partial\varphi = \epsilon\sqrt{[-(\partial b/\partial\varphi)^2]}\, b''_\rho \tag{3.5}$$

and we use the general expansion

$$y_\nu = y \cdot u u_\nu - y \cdot b b_\nu - y \cdot b' b'_\nu - y \cdot b'' b''_\nu \tag{3.6}$$

to express

$$\dot{b}_\nu = -b \cdot w u_\nu - \dot{b} \cdot b' b'_\nu - \dot{b} \cdot b'' b''_\nu, \tag{3.7}$$

$$w_\nu = -b \cdot w b_\nu - b' \cdot w b'_\nu - b'' \cdot w b''_\nu \tag{3.8}$$

We find

$$d\sigma_\mu = \epsilon^2[(1 - \epsilon b \cdot w)b_\mu - \epsilon b \cdot w u_\mu]\, d\Omega\, d\tau \tag{3.9}$$

which has a component along u due to the fact that b is the normal to the world line at the vertex of the cone, not the normal to the surface in this case. We use (2.5), (2.6), (2.18), (2.19), (2.20), and (3.9) to derive

$$\begin{aligned}\Theta_{\mu\nu}\, d\sigma_\nu = (e^2/16\pi^2\epsilon^2)\{&[(1 - \epsilon b \cdot w)^2 \\ &+ \epsilon^2 w^2](u_\mu + b_\mu) - (1 - \epsilon b \cdot w)[(1 + \epsilon b \cdot w)u_\mu \\ &+ \epsilon b \cdot w b_\mu] + \epsilon[(1 - \epsilon b \cdot w)b \cdot w(u_\mu + b_\mu) \\ &+ (1 - 2\epsilon b \cdot w)w_\mu] - \tfrac{1}{2}[(1 - \epsilon b \cdot w)b_\mu \\ &- \epsilon b \cdot w u_\mu]\}\end{aligned} \tag{3.10}$$

To perform the angular integrations, we need the relation

$$\int b_\mu b_\nu\, d\Omega = -\tfrac{4}{3}\pi(g_{\mu\nu} - u_\mu u_\nu) \tag{3.11}$$

while a similar expression with three factors of b vanishes. The result is

$$\int_{\Sigma'} \Theta_{\mu\nu}\, d\sigma_\nu = (e^2/4\pi\epsilon) \int_{\tau_1}^{\tau_2} d\tau(-\tfrac{1}{2}w_\mu + \tfrac{2}{3}\epsilon w^2 u_\mu) \tag{3.12}$$

which can be rewritten as

$$-\int_{\Sigma'} \Theta_{\mu\nu}\, d\sigma_\nu = P_\mu{}^C(\tau_2) - P_\mu{}^C(\tau_1) + P_\mu{}^R(\tau_1, \tau_2) \tag{3.13}$$

Equation (3.12) is an exact expression, not an approximation for small ϵ. The reason the term in $\dot{w}_\mu$ is absent is that the fields on the tube are referred to the proper time at the vertex of the light-cone instead of that corresponding to the same coordinate time in the instantaneous rest frame. Since the field originates at the retarded time, the former approach is more consistent, and it agrees with our definition of the self-energy. It is also true that the distance from the world line is measured differently and the surfaces are not exactly the same, but the flux should be the same.

The same procedure is used by Rowe (1974), where the difference between his and Dirac's result is attributed to changes in the end of the tube. This is to some extent a matter of interpretation.

Furthermore, equation (3.12) is also derived in a paper by Hogan (1973), which was followed by a recantation (see footnote 2). He uses it to derive the usual Lorentz-Dirac equation.

[2] Hogan (1974). The surface at infinity does not appear to matter to the argument in the paper (Hogan, 1973). We note that we obtain the correct radiation rate if we let $\epsilon \to \infty$.

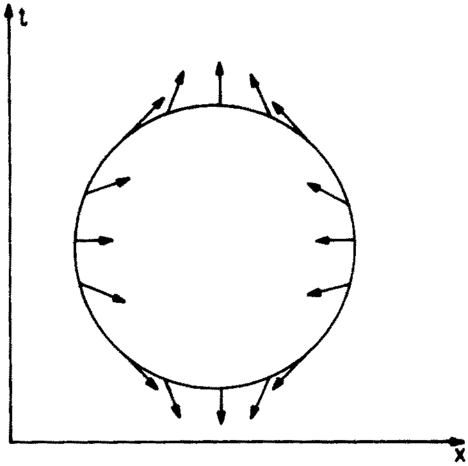

Figure 2—Selection of the normal to a closed surface in Minkowski space to apply the generalization of the divergence theorem $\int \Theta_{\mu\nu,\,\nu}\, d^4x = \oint \Theta_{\mu\nu}\, d\sigma_\nu$.

The application of the divergence theorem in Minkowski space requires special care. As shown in Figure 2, we can take them pointing outward for timelike normals, inward for spacelike ones and along the appropriate tangent for lightlike "normals."

4. *The Equation of Motion*

If we apply equation (3.12) to an infinitesimal interval $d\tau$, it gives us the rate at which electromagnetic energy-momentum is lost by the particle. The Coulomb part is incorporated in the rate of change of the mechanical energy-momentum p_μ, and we obtain the equation of motion

$$dp_\mu/d\tau - (e^2/6\pi)w^2 u_\mu = F_\mu \tag{4.1}$$

instead of the Lorentz-Dirac equation (1.1).

If the force is of electromagnetic origin, it has the form

$$F_\mu = -eF_{\mu\nu}u_\nu \tag{4.2}$$

and it satisfies

$$F_\mu u_\mu \equiv 0 \tag{4.3}$$

This implies that contraction of the left-hand side of equations (1.1) and (4.1) with u_μ has to give a zero result. This is the case with the Lorentz-Dirac equation when the derivative of equation (2.18) is used to show that

$$\Gamma_\mu u_\mu \equiv 0 \tag{4.4}$$

In other words, the four equations (one for each value of μ) are not independent, a reflection of the three degrees of freedom of a free point particle, irrespective of speed. The equation for $\mu = 0$ normally expresses the conservation of energy, especially if, for the Lorentz-Dirac equation, the Schott term is included (Teitelboim, 1970a) in

$$p_\mu = mu_\mu - (e^2/6\pi)w_\mu \tag{4.5}$$

We have no Schott term in our derivation of equation (4.1), and we are very reluctant to introduce it arbitrarily at this stage. It is also hard to justify giving up equation (4.3) by the required introduction of a nonelectromagnetic force that would have to feed the radiated energy to the particle. Thus, if we insist on keeping the definition

$$p_\mu = mu_\mu \tag{4.6}$$

we are forced to conclude that the rest mass m is not a constant. Instead, equation (4.1) leads to the relationship

$$\dot{m} = (e^2/6\pi)w^2 \tag{4.7}$$

that is, the mass itself feeds the radiation energy. We note that the four-acceleration is a spacelike vector, and that w^2 is always negative for an accelerated particle.

The Lorentz-Dirac equation with an incident radiation field is invariant under time reversal (Rohrlich, 1965). The modified equation (4.1) is not; in particular, m is a decreasing function of time, and the time-reversed solution would correspond to an increasing mass. It is not clear, though, what the physical importance of time reversal invariance is. Even when equations are invariant, solutions need not be due to the boundary conditions. If a retarded interaction between charged particles is postulated, the invariance is also lost. In scattering of a classical electromagnetic field by a macroscopic object, the solution reduces to outgoing spherical waves; although the time-reversed solution can be written down and satisfies Maxwell's equations, it would be practically impossible to produce such an incoming spherical wave in the laboratory. Similar arguments apply to wave packets in quantum mechanics.

The situation might be different where strong time reflection is concerned, that is, when particles and antiparticles are interchanged, too. Antiparticles can be considered to be propagating backwards in time and they might radiate in a time-reversed pattern.

There is no evidence that the mass of a particle changes due to radiation, but this disagreement with experiments might disappear when the theory is quantized. This would be similar to the lack of radiation from the hydrogen atom, which was only explained by the quantum theory. Furthermore, it is far from established that a single charged particle radiates and whether a mass can be defined for a particle interacting with others.

The variable mass is a disturbing feature of this equation, but not more so than asymptotic conditions, nonlocal interactions, and preacceleration.

Substituting equation (4.7) back into (4.1), it reduces to

$$mw_\mu = F_\mu \tag{4.8}$$

which is the usual equation of motion without radiation reaction when m is constant. But here it is supplemented by equation (4.7), which in fact represents a new degree of freedom for the classical charged particle.

5. *Concluding Remarks*

We have shown that the balance of energy and momentum for a charged particle does not necessarily lead to the Lorentz-Dirac equation, but that a minor modification of the usual argument leads to the elimination of the Schott term. The Coulomb energy-momentum was defined by integration over the light cone instead of a spacelike hyperplane. Only the retarded fields were used, because they are closer to our intuition, and the advanced fields are not crucial to the argument.

We used a Lorentz gauge and the Liénard-Wiechert potentials, although there are indications that the Coulomb or radiation gauge is more physically significant (Marx, 1970). But in the Lorentz gauge, both the potentials and the fields depend only on one point on the world line, and the fields are gauge invariant.

If we eliminate the Schott term from the equation of motion, we are lead to the assumption of a variable rest mass for the particle. The decrease of the mass would account for the emitted radiation. This would probably be an effect that would disappear in a quantized form of the theory. If the world line is allowed to turn back in time, the mass could also increase.

The nature of the force was not discussed in this paper, beyond the suggestion that it can be electromagnetic in nature. In extending this approach to several charged particles, the natural choice would be the retarded electromagnetic interaction. The variable mass would avoid the objectionable result of the head-on collision of two equal particles, which radiate at the same time they gain kinetic energy if the radiation reaction term is ignored (Huschilt, et al., 1973).

Another puzzling consequence of the Lorentz-Dirac equation is no longer true for the new approach which is the vanishing of the Abraham vector for uniformly accelerated motion, as a result of which the motion of a neutral particle would be the same as that of a charged particle, in spite of the radiation emitted by the latter.

It is necessary to explore further the effects of the elimination of the Schott term to decide which equation is better, although the results are certain to be modified by quantum mechanics in either case. Actually, the possible use in a quantized theory may be considered the final test on the validity of such an equation.

Acknowledgments

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